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A balanced canonical form for discrete-time minimal systems using characteristic maps

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Abstract

This paper presents a balanced canonical form for the class of discrete-time minimal systems. The main tool is to establish a bijection between the set of minimal systems and the class of minimal discrete-time asymptotically stable systems of the same dimension. This characteristic map is shown to preserve system equivalence and balancing. The canonical form for discrete-time minimal systems is then constructed by mapping the system to its discrete-time asymptotically stable counterpart via the characteristic map, transforming the resulting system to Lyapunov-balanced canonical form, and returning to the original system class by means of the inverse characteristic map. © 1998 Elsevier Science Inc. All rights reserved.

1. Introduction

The aim of this paper is to present a balanced canonical form for the class $L_n^{p,m}$ of discrete-time minimal systems $(A, B, C, D) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m} \times \mathbb{K}^{p \times n} \times \mathbb{K}^{p \times m}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In the work of Ober [1] on balanced canonical forms for different classes of transfer functions, the emphasis is mainly on the continuous-time case, and discrete-time canonical forms for the set of minimal discrete-time asymptotically stable systems $D_n^{p,m} \subset L_n^{p,m}$ as well as for bounded real and positive real systems are obtained by relating the discrete-time function classes to their continuous-time counterparts via the Möbius transform. This approach obviously fails for systems in $L_n^{p,m}$, since the Möbius transform is

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not a bijection between the classes of minimal systems with time axis \mathbb{R} and \mathbb{Z} , respectively.

Our approach is motivated by the work of Fuhrmann and Ober [2,3] on diffeomorphisms between sets of continuous-time linear systems. Using the Youla–Kucera parametrization of all stabilizing controllers, the authors investigate a canonical pair of solutions of the Bezout equations to obtain the associated characteristic functions; moreover, state-space formulas for the different characteristic maps are derived.

Our solution of the canonical form problem is to establish a bijective correspondence between systems in $L_n^{p,m}$ and systems in $D_n^{p,m}$ via the characteristic map, which is defined such that it respects balancing. Hence a balanced canonical form can be obtained as follows: First, the given system is mapped to its asymptotically stable counterpart via the characteristic map; the resulting system is brought into Lyapunov-balanced canonical form and then transformed back to the original system class via the inverse characteristic map. Observe that since the characteristic map is given in state-space terms, this method together with any algorithm for the determination of the balanced canonical form of a system in $D_n^{p,m}$ constitutes an implementable method for the calculation of balanced canonical forms.

The paper is structured as follows: Section 2 provides some basic facts about balancing and canonical forms. An important tool is derived in Section 3: the discrete-time analogue of the Bucy relations [4]. These are well-known in the continuous-time case but – to the authors’ best knowledge – have not yet been presented for discrete-time systems. The characteristic map and its inverse are introduced in Section 4, and the most important properties of these two maps are discussed. The proofs in that section involve calculations which are tedious but not difficult. For the sake of completeness, they have been included in Appendix A. Finally, Section 5 treats the examination of the characteristic map with respect to balancing, and a balanced canonical form for discrete-time minimal systems is proposed.

2. Preliminaries

The class of minimal systems $(A, B, C, D) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m} \times \mathbb{K}^{p \times n} \times \mathbb{K}^{p \times m}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, is denoted by $L_n^{p,m}$. The system (A, B, C, D) , or simply, the matrix A , is said to be (discrete-time asymptotically) stable iff all the eigenvalues of A lie in the open unit disk of the complex plane. The set $D_n^{p,m} \subset L_n^{p,m}$ consists of all minimal systems which are discrete-time asymptotically stable. Two types of discrete-time balanced systems will be considered: Lyapunov-balanced and linear quadratic Gaussian-balanced systems (LQG).

Lyapunov-balanced systems: For a system $(A, B, C, D) \in D_n^{p,m}$, the controllability Lyapunov equation reads

$$APA^* - P + BB^* = 0.$$

The asymptotic stability of A guarantees the existence of a unique solution P , which is called controllability Gramian of (A, B, C, D) . Dually, the observability Lyapunov equation is

$$A^*QA - Q + C^*C = 0$$

and its unique solution Q is called observability Gramian. Minimality of (A, B, C, D) implies that $P > 0$ and $Q > 0$. A system $(A, B, C, D) \in D_n^{p,m}$ is said to be Lyapunov-balanced iff $P = Q = \Sigma > 0$, a real diagonal matrix. Σ is called Lyapunov–Gramian of (A, B, C, D) .

LQG-balanced systems: The discrete-time generalized control ARE (algebraic Riccati equation) associated to $(A, B, C, D) \in L_n^{p,m}$ is given by

$$A^*XA - X + C^*C = (A^*XB + C^*D)(I + D^*D + B^*XB)^{-1}(D^*C + B^*XA).$$

Its dual counterpart is the discrete-time generalized filter ARE

$$AYA^* - Y + BB^* = (AYC^* + BD^*)(I + DD^* + CYC^*)^{-1}(DB^* + CYA^*).$$

Solutions X and Y are said to be stabilizing iff $A - BF$ and $A - HC$, where

$$F = (I + D^*D + B^*XB)^{-1}(D^*C + B^*XA) \text{ and } H = (AYC^* + BD^*)(I + DD^* + CYC^*)^{-1},$$

are discrete-time asymptotically stable.

In the following section, we show that these equations can be put in a standard form from which it can be deduced that for a minimal system, there exist uniquely determined stabilizing solutions $X > 0$ and $Y > 0$ to these ARE's [5]. A system (A, B, C, D) is called LQG-balanced iff there exists a real diagonal matrix $\Sigma > 0$ such that $X = Y = \Sigma$, the LQG-Gramian of (A, B, C, D) .

Balancing and system equivalence: The controllability and observability Gramians \hat{P} and \hat{Q} of a system $(TAT^{-1}, TB, CT^{-1}, D)$ are related to the corresponding Gramians P and Q of (A, B, C, D) via

$$\hat{P} = TPT^*, \quad \hat{Q} = T^{-1*}QT^{-1}.$$

For every pair of positive definite matrices P and Q , there exists a nonsingular matrix T such that

$$TPT^* = T^{-1*}QT^{-1} = \Sigma,$$

with $\Sigma > 0$ a real diagonal matrix [5]. Thus, every system in $D_n^{p,m}$ has an equivalent system which is Lyapunov-balanced. Since the eigenvalues of the product of the Gramians are invariant under system equivalence, equivalent Lyapunov-balanced systems have the same Lyapunov–Gramian Σ if we require its diagonal entries to be ordered, i.e.,

$$\Sigma = \text{diag}(\sigma_1 I_{n_1}, \dots, \sigma_k I_{n_k}),$$

with $\sigma_1 > \dots > \sigma_k > 0$ and $\sum_{i=1}^k n_i = n$. The values $\sigma_1, \dots, \sigma_k$ are called Hankel singular values of the equivalence class $\{(TAT^{-1}, TB, CT^{-1}, D), T \in \text{GL}_n(\mathbb{K})\}$. If (A, B, C, D) is Lyapunov-balanced with Lyapunov–Gramian Σ as above, all equivalent balanced systems are of the form (SAS^*, SB, CS^*, D) with $S = \text{diag}(S_1, \dots, S_k)$ and $S_i \in \mathbb{K}^{n_i \times n_i}$ unitary for $1 \leq i \leq k$.

Similarly, the solutions \hat{X} and \hat{Y} of the algebraic Riccati equations associated to a system $(TAT^{-1}, TB, CT^{-1}, D)$ are connected with the solutions X and Y corresponding to (A, B, C, D) by

$$\hat{X} = T^{-1*}XT^{-1}, \quad \hat{Y} = TYT^*.$$

Thus every system in $L_n^{p,m}$ possesses an equivalent LQG-balanced system. The diagonal entries of the LQG-Gramian $\Sigma = \text{diag}(\sigma_1 I_{n_1}, \dots, \sigma_k I_{n_k})$ are called LQG-singular values and they are invariant under system equivalence. If (A, B, C, D) is LQG-balanced, the class of equivalent LQG-balanced systems is again of the form (SAS^*, SB, CS^*, D) , where $S = \text{diag}(S_1, \dots, S_k)$ with unitary matrices $S_i \in \mathbb{K}^{n_i \times n_i}$.

Canonical forms: Let \sim denote an equivalence relation on a set M . Consider a map $\phi: M \rightarrow M$ with the following properties: $m_1 \sim m_2$ implies $\phi(m_1) = \phi(m_2)$ and $m \sim \phi(m)$ for all $m \in M$. Then the induced map $\phi_{\text{ind}}: M/\sim \rightarrow M$, $[m] \mapsto \phi(m)$ is well-defined and it assigns a distinguished representative $\phi(m)$ to every equivalence class $[m] = \{m_1 \in M, m \sim m_1\}$. Such a map ϕ is called a canonical form for M/\sim . This formal concept will be applied to $M = L_n^{p,m}$ and $M = D_n^{p,m}$, where \sim denotes the equivalence relation defined by system equivalence.

3. Discrete-time Bucy relation for minimal systems

Consider the dual pair of discrete-time algebraic Riccati equations

$$X = A^*XA + C^*C - (A^*XB + C^*D)(I + D^*D + B^*XB)^{-1}(D^*C + B^*XA), \quad (1)$$

$$Y = AYA^* + BB^* - (AYC^* + BD^*)(I + DD^* + CYC^*)^{-1}(DB^* + CYA^*). \quad (2)$$

Applying a result by Wimmer [6], we can rewrite (1) in standard form:

$$X = \hat{A}^*X\hat{A} + \hat{Q} - \hat{A}^*X\hat{B}(I + \hat{B}^*X\hat{B})^{-1}\hat{B}^*X\hat{A}, \quad (3)$$

where with $R := I + D^*D > 0$,

$$\hat{A} := A - BR^{-1}D^*C, \quad \hat{B} := BR^{-1/2}, \quad \hat{Q} := C^*(I - DR^{-1}D^*)C.$$

Analogously, the standard form of (2) is given by

$$Y = \bar{A}Y\bar{A}^* + \bar{Q} - \bar{A}Y\bar{C}^*(I + \bar{C}Y\bar{C}^*)^{-1}\bar{C}Y\bar{A}^*, \quad (4)$$

where for $S := I + DD^* > 0$,

$$\bar{A} := A - BD^*S^{-1}C, \quad \bar{C} := S^{-1/2}C, \quad \bar{Q} := B(I - D^*S^{-1}D)B^*.$$

These transformations leave the set of stabilizing solutions $X > 0$ and $Y > 0$ of the algebraic Riccati equations (1) and (2) invariant.

Lemma 3.1. *The ARE's (3) and (4) can be rewritten in the form*

$$X = \bar{A}^*X\bar{A} + \bar{C}^*\bar{C} - \bar{A}^*X\bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*X\bar{A}, \quad (5)$$

$$Y = \bar{A}Y\bar{A}^* + \bar{B}\bar{B}^* - \bar{A}Y\bar{C}^*(I + \bar{C}Y\bar{C}^*)^{-1}\bar{C}Y\bar{A}^*, \quad (6)$$

where $\bar{B} := \hat{B}$ from above.

Proof. It is easy to see that $DR = SD$, hence $D^*S^{-1} = R^{-1}D^*$ and $\bar{A} = \hat{A}$. Furthermore,

$$\hat{Q} = C^*(I - DR^{-1}D^*)C = C^*(I - DD^*S^{-1})C = C^*S^{-1}C = \bar{C}^*\bar{C}$$

and similarly, $\bar{Q} = \bar{B}\bar{B}^*$. \square

Hence after transforming the two initial ARE's separately, we have obtained again a dual pair of algebraic Riccati equations. Define now:

$$F := (I + D^*D + B^*XB)^{-1}(D^*C + B^*XA), \quad (7)$$

$$H := (AYC^* + BD^*)(I + DD^* + CYC^*)^{-1}, \quad (8)$$

$$\bar{F} := (I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*X\bar{A}, \quad (9)$$

$$\bar{H} := \bar{A}Y\bar{C}^*(I + \bar{C}Y\bar{C}^*)^{-1}. \quad (10)$$

Then $A - BF$, $\bar{A} - \bar{B}\bar{F}$, $A - HC$ and $\bar{A} - \bar{H}\bar{C}$ are discrete-time asymptotically stable and the following computation shows that

$$A - BF = \bar{A} - \bar{B}\bar{F} \quad \text{and} \quad A - HC = \bar{A} - \bar{H}\bar{C}. \quad (11)$$

Define

$$W := (I + D^*D + B^*XB)^{-1} = (R + B^*XB)^{-1},$$

then

$$\begin{aligned} \bar{A} - \bar{B}\bar{F} &= \bar{A} - \bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*X\bar{A} \\ &= A - BR^{-1}D^*C - B(R + B^*XB)^{-1}B^*X(A - BR^{-1}D^*C) \\ &= A - BR^{-1}D^*C - BWB^*X(A - BR^{-1}D^*C) \\ &= A - BWB^*XA - BW(W^{-1} - B^*XB)R^{-1}D^*C \\ &= A - BW(B^*XA + D^*C) = A - BF \end{aligned}$$

and analogously $\bar{A} - \bar{H}\bar{C} = A - HC$. Rewriting

$$\begin{aligned}\bar{A} - \bar{B}\bar{F} &= \bar{A} - \bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*X\bar{A} \\ &= (I - \bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*X)\bar{A} \\ &= (I - \bar{B}\bar{B}^*X(I + \bar{B}\bar{B}^*X)^{-1})\bar{A} = (I + \bar{B}\bar{B}^*X - \bar{B}\bar{B}^*X)(I + \bar{B}\bar{B}^*X)^{-1}\bar{A} \\ &= (I + \bar{B}\bar{B}^*X)^{-1}\bar{A}\end{aligned}\quad (12)$$

and analogously,

$$\bar{A} - \bar{H}\bar{C} = \bar{A}(I + Y\bar{C}^*\bar{C})^{-1}, \quad (13)$$

we obtain the relation

$$(I + \bar{B}\bar{B}^*X)(\bar{A} - \bar{B}\bar{F}) = (\bar{A} - \bar{H}\bar{C})(I + Y\bar{C}^*\bar{C})$$

for the solutions X and Y of (1) and (2).

Observe that the intertwining matrices in the above relation are invertible; we like to stress that this result is obtained without making any assumptions on the invertibility of A , $A - BF$ or $A - HC$. In the sequel we will show that $A - BF$ and $A - HC$ are indeed similar.

Theorem 3.2 (Discrete-time Bucy relation for minimal systems). *Let $(A, B, C, D) \in L_n^{p,m}$, let $X > 0$ and $Y > 0$ denote the stabilizing solutions of the Riccati equations (1) and (2) and let F and H be defined as in (7) and (8). Then*

$$(I + YX)(A - BF) = (A - HC)(I + YX). \quad (14)$$

Proof. We use the transformed Riccati equations in order to prove the assertion for $\bar{A} - \bar{B}\bar{F}$ and $\bar{A} - \bar{H}\bar{C}$, which in view of relations (11) yields the result. Let us start with the analysis of Eq. (5):

$$\begin{aligned}X &= \bar{A}^*X\bar{A} + \bar{C}^*\bar{C} - \bar{A}^*X\bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*X\bar{A} \\ &= \bar{A}^*X(\bar{A} - \bar{B}\bar{F}) + \bar{C}^*\bar{C}.\end{aligned}\quad (15)$$

Thus

$$I + YX = Y\bar{A}^*X(\bar{A} - \bar{B}\bar{F}) + (I + Y\bar{C}^*\bar{C}).$$

Now in view of (13) we get

$$(\bar{A} - \bar{H}\bar{C})(I + YX) = (\bar{A} - \bar{H}\bar{C})Y\bar{A}^*X(\bar{A} - \bar{B}\bar{F}) + \bar{A}. \quad (16)$$

Next, we proceed analogously with the dual Riccati equation (6), i.e.,

$$\begin{aligned}Y &= \bar{A}Y\bar{A}^* + \bar{B}\bar{B}^* - \bar{A}Y\bar{C}^*(I + \bar{C}Y\bar{C}^*)^{-1}\bar{C}Y\bar{A}^* \\ &= (\bar{A} - \bar{H}\bar{C})Y\bar{A}^* + \bar{B}\bar{B}^*.\end{aligned}\quad (17)$$

There holds

$$I + YX = (\bar{A} - \bar{H}\bar{C})Y\bar{A}^*X + (I + \bar{B}\bar{B}^*X).$$

Looking at (12) this results in

$$(I + YX)(\bar{A} - \bar{B}\bar{F}) = (\bar{A} - \bar{H}\bar{C})Y\bar{A}^*X(\bar{A} - \bar{B}\bar{F}) + \bar{A}. \quad (18)$$

Observe that the right-hand sides of (16) and (18) coincide, which means that

$$(I + YX)(\bar{A} - \bar{B}\bar{F}) = (\bar{A} - \bar{H}\bar{C})(I + YX). \quad (19)$$

Finally, note that $I + YX = Y(Y^{-1} + X)$ with $Y > 0$ and $Y^{-1} + X > 0$, which implies that $I + YX$ is invertible. \square

4. The characteristic map and its inverse

In this section we define two maps

$$\psi_L: L_n^{p,m} \rightarrow D_n^{p,m} \quad \text{and} \quad I_{\psi_L}: D_n^{p,m} \rightarrow L_n^{p,m}$$

which preserve system equivalence, minimality and balancing. These properties are consequences of close relations between the ARE's and Lyapunov equations for corresponding systems in $L_n^{p,m}$ and $D_n^{p,m}$, respectively. In fact, the map ψ_L is precisely the discrete-time analogue of the characteristic map χ_L defined by Ober and Fuhrmann [3].

Definition 4.1. Let $(A, B, C, D) \in L_n^{p,m}$ and let $X > 0$ and $Y > 0$ be the stabilizing solutions of the Riccati equations (1) and (2). Then the system

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} := \psi_L \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \\ := \left(\begin{array}{c|c} A - B(R + B^*XB)^{-1}(B^*XA + D^*C) & B(R + B^*XB)^{-1/2} \\ \hline (S + CYC^*)^{-1/2}C(I + YX) & D \end{array} \right)$$

is called the L -characteristic of the system (A, B, C, D) .

The subsequent lemma shows that the L -characteristic map transforms a system with no restrictions on the pole location to a stable system of the same McMillan degree. The proofs of this and other results from this section are collected in Appendix A.

Lemma 4.2. *The L -characteristic of a minimal system is discrete-time asymptotically stable and minimal. The L -characteristics of two equivalent systems are equivalent; moreover, the corresponding transformation matrices are the same, i.e., $\psi_L \circ T = T \circ \psi_L$, where T in $\text{GL}_n(\mathbb{K})$ denotes the map which assigns an equivalent system to a system in $L_n^{p,m}$ or $D_n^{p,m}$ in the following way:*

$$T(A, B, C, D) := (TAT^{-1}, TB, CT^{-1}, D).$$

The following property of the characteristic map is crucial with respect to balancing: The solutions of the Lyapunov equations associated to the L -characteristic system can be expressed in terms of the stabilizing solutions of the Riccati equations (5) and (6); actually, it is this fact that has provided the motivation for introducing the L -characteristic as in Definition 4.1.

Proposition 4.3. *Let $(A, B, C, D) \in L_n^{p,m}$ and let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in D_n^{p,m}$ be its L -characteristic system, with X and Y the stabilizing solutions of the associated Riccati equations. Then the Lyapunov equations*

$$P = \mathcal{A}P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*, \quad (20)$$

$$Q = \mathcal{A}^*Q\mathcal{A} + \mathcal{C}^*\mathcal{C}, \quad (21)$$

have the unique solutions

$$P = Y(I + XY)^{-1} = (I + YX)^{-1}Y, \quad (22)$$

$$Q = X(I + YX) = (I + XY)X. \quad (23)$$

The next step in our analysis is to define a map from $D_n^{p,m}$ to $L_n^{p,m}$, namely the inverse L -characteristic, which is such that the stabilizing solutions of the Riccati equations associated to the image system under this map can be calculated in terms of the solutions of the Lyapunov equations of the original system. Indeed we will see that the inverse L -characteristic is really the inverse mapping of the L -characteristic.

Definition 4.4. Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in D_n^{p,m}$ and let P and Q be the solutions of the Lyapunov equations

$$P = \mathcal{A}P\mathcal{A}^* + \mathcal{B}\mathcal{B}^*,$$

$$Q = \mathcal{A}^*Q\mathcal{A} + \mathcal{C}^*\mathcal{C}.$$

Then the system

$$I_{\psi_L} \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right) := \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right)$$

with

$$D_1 := \mathcal{D},$$

$$C_1 := S^{1/2}(I - \mathcal{C}P(I + QP)^{-1}\mathcal{C}^*)^{-1/2}\mathcal{C}(I + PQ)^{-1},$$

$$B_1 := \mathcal{B}(I - \mathcal{B}^*(I + QP)^{-1}Q\mathcal{B})^{-1/2}R^{1/2},$$

$$A_1 := \mathcal{A} + \mathcal{B}(I - \mathcal{B}^*(I + QP)^{-1}Q\mathcal{B})^{-1}\mathcal{B}^*(I + QP)^{-1}Q\mathcal{A} + B_1R^{-1}\mathcal{D}^*C_1,$$

where $R := I + \mathcal{D}^* \mathcal{D}$, $S := I + \mathcal{D} \mathcal{D}^*$, is called the *inverse L-characteristic* of the system $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$.

For well-definedness of I_{ψ_L} , we need that the matrix $I - \mathcal{B}^*(I + QP)^{-1}Q\mathcal{B}$ is positive definite. This follows from

$$0 < (Q^{-1} + \mathcal{A}P\mathcal{A}^*)^{-1} = (Q^{-1} + P - \mathcal{B}\mathcal{B}^*)^{-1} = (I + QP - Q\mathcal{B}\mathcal{B}^*)^{-1}Q$$

and

$$(I - \mathcal{B}^*(I + QP)^{-1}Q\mathcal{B})^{-1} = I + \mathcal{B}^*(I + QP - Q\mathcal{B}\mathcal{B}^*)^{-1}Q\mathcal{B}.$$

The matrix $I - \mathcal{C}P(I + QP)^{-1}\mathcal{C}^*$ is treated analogously. The fact that the inverse characteristic system of a stable minimal system is minimal, i.e., $I_{\psi_L}(D_n^{p,m}) \subseteq L_n^{p,m}$, follows by the same argumentation as in the proof of Lemma 4.2.

First, we investigate the relation between the solutions of the Lyapunov equations associated with a system in $D_n^{p,m}$ and the ARE's for its inverse characteristic system.

Proposition 4.5. *Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in D_n^{p,m}$. Let P and Q be the solutions of the Lyapunov equations (20) and (21). Then the ARE's associated to $(A_1, B_1, C_1, D_1) = I_{\psi_L}(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$, i.e.,*

$$X = A_1^*XA_1 + C_1^*C_1 - (A_1^*XB_1 + C_1^*D_1)(R_1 + B_1^*XB_1)^{-1}(D_1^*C_1 + B_1^*XA_1), \quad (24)$$

$$Y = A_1YA_1^* + B_1B_1^* - (A_1YC_1^* + B_1D_1^*)(S_1 + C_1YC_1^*)^{-1}(D_1B_1^* + C_1YA_1^*), \quad (25)$$

where $R_1 = I + D_1^*D_1$, $S_1 = I + D_1D_1^*$, have the unique solutions

$$X = Q(I + PQ)^{-1} = (I + QP)^{-1}Q, \quad (26)$$

$$Y = P(I + QP) = (I + PQ)P. \quad (27)$$

We proceed by showing that the inverse L -characteristic map is really the inverse mapping of the L -characteristic ψ_L . This is accomplished by proving the following two propositions.

Proposition 4.6. $I_{\psi_L} \circ \psi_L: L_n^{p,m} \rightarrow L_n^{p,m}$ is the identity map, i.e., the characteristic map ψ_L is injective.

Proposition 4.7. $\psi_L \circ I_{\psi_L}: D_n^{p,m} \rightarrow D_n^{p,m}$ is the identity map, i.e., the characteristic map ψ_L is surjective.

In view of Lemma 4.2, it is now a straightforward consequence of $I_{\psi_L} = \psi_L^{-1}$ that also I_{ψ_L} respects system equivalence.

Corollary 4.8. *The inverse L -characteristic systems of two equivalent systems are equivalent; moreover*

$$I_{\psi_L} \circ T = T \circ I_{\psi_L}$$

for $T \in \text{GL}_n(\mathbb{K})$ and $T(A, B, C, D) = (TAT^{-1}, TB, CT^{-1}, D)$.

Remark 4.9. Let $\mathbb{K} = \mathbb{R}$ and let $L_{n,0}^{p,m} \subset L_n^{p,m}$ denote the set of minimal systems with zero feedthrough term. Analogously, let $D_{n,0}^{p,m}$ denote the set of strictly causal systems in $D_n^{p,m}$. The system class $L_{n,0}^{p,m}$ can be interpreted as an open subset of the Euclidean space $\mathbb{R}^{n^2+nm+pm}$ in a natural way. Note that for $D = 0$, the ARE's (1) and (2) are already in the standard form of (5) and (6). According to a result by Delchamps [7], the stabilizing solution to an ARE is an analytic function of the system matrices. The same is true for the solutions of Lyapunov equations for systems in $D_n^{p,m}$. Let \sim denote the equivalence relation defined by system equivalence. Following the argumentation by Ober and Fuhrmann [3], the map ψ_L constitutes a diffeomorphism between the sets $L_{n,0}^{p,m} / \sim$ and $D_{n,0}^{p,m} / \sim$.

5. Balanced canonical form

A canonical form for the set $L_n^{p,m} / \sim$ is a map which assigns a distinguished representative of the equivalence class

$$\{(TAT^{-1}, TB, CT^{-1}, D), T \in \text{GL}_n(\mathbb{K})\}$$

to $(A, B, C, D) \in L_n^{p,m}$.

If (A, B, C, D) is LQG-balanced with LQG-Gramian $\Sigma = \text{diag}(\sigma_1 I_{n_1}, \dots, \sigma_k I_{n_k})$, then the LQG-balanced systems which are equivalent to (A, B, C, D) are precisely those of the form (SAS^*, SB, CS^*, D) with $S = \text{diag}(S_1, \dots, S_k)$ and $S_i \in \mathbb{K}^{n_i \times n_i}$ unitary for $1 \leq i \leq k$. Thus, a balanced canonical form is a map which assigns a distinguished representative of the equivalence class

$$\{(SAS^*, SB, CS^*, D), S = \text{diag}(S_1, \dots, S_k), S_i \in \mathbb{K}^{n_i \times n_i} \text{ unitary}\},$$

to the balanced system (A, B, C, D) . The integers k and n_1, \dots, n_k with $\sum_{i=1}^k n_i = n$ are determined by the block partition of the Gramian Σ of (A, B, C, D) .

The analysis of the characteristic map from the point of view of balancing requires a minor modification. Define a new characteristic map for a system $(A, B, C, D) \in L_n^{p,m}$ by

$$\tilde{\psi}_L \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) := \left(\begin{array}{c|c} T^{1/4} \mathcal{A} T^{-1/4} & T^{1/4} \mathcal{B} \\ \hline \mathcal{C} T^{-1/4} & \mathcal{D} \end{array} \right),$$

where $T := (I + YX)^*(I + YX)$ and X, Y are the stabilizing solutions of (1) and (2), i.e., $\tilde{\psi}_L(A, B, C, D)$ is equivalent to $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) = \psi_L(A, B, C, D)$.

Similarly, for $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in D_n^{p,m}$, where P and Q are the associated solutions of the Lyapunov equations, define $U := (I + PQ)^*(I + PQ)$ and

$$\tilde{I}_{\psi_L} \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right) := \left(\begin{array}{c|c} U^{-1/4} A_1 U^{1/4} & U^{-1/4} B_1 \\ \hline C_1 U^{1/4} & D_1 \end{array} \right)$$

where $(A_1, B_1, C_1, D_1) = I_{\psi_L}(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$.

Note that $\tilde{I}_{\psi_L} \circ \psi_L$ is not the identity map, but still $\tilde{I}_{\psi_L} \tilde{\psi}_L(A, B, C, D) \sim (A, B, C, D)$ for all $(A, B, C, D) \in L_n^{p,m}$.

Theorem 5.1. *If $(A, B, C, D) \in L_n^{p,m}$ is discrete-time LQG-balanced with Gramian Σ , then $\tilde{\psi}_L(A, B, C, D)$ in $D_n^{p,m}$ is Lyapunov-balanced with Gramian Σ . Conversely, if $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in D_n^{p,m}$ is Lyapunov-balanced with Gramian Σ , then $\tilde{I}_{\psi_L}(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in L_n^{p,m}$ is discrete-time LQG-balanced with Gramian Σ .*

If ϕ_D is a Lyapunov-balanced canonical form for $D_n^{p,m} / \sim$, then $\phi_L := \tilde{I}_{\psi_L} \circ \phi_D \circ \tilde{\psi}_L$ is a discrete-time LQG-balanced canonical form for $L_n^{p,m} / \sim$.

Proof. Let (A, B, C, D) be minimal, and let X, Y be as introduced above. By Proposition 4.3 the two Lyapunov equations associated to

$$\left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right) = \psi_L \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$$

are solved by $P = (I + YX)^{-1}Y$ and $Q = (I + XY)X$. Moreover, the solutions to the Lyapunov equations associated to $\tilde{\psi}_L(A, B, C, D)$ are given by

$$\begin{aligned} P_1 &= T^{1/4} P T^{1/4} = ((I + YX)^*(I + YX))^{1/4} (I + YX)^{-1} Y ((I + YX)^*(I + YX))^{1/4}, \\ Q_1 &= T^{-1/4} Q T^{-1/4} = ((I + YX)^*(I + YX))^{-1/4} (I + XY) X ((I + YX)^*(I + YX))^{-1/4}. \end{aligned}$$

If (A, B, C, D) is LQG-balanced, then $Y = X = \Sigma$ is diagonal, and therefore

$$P_1 = Y = X = Q_1 = \Sigma,$$

i.e., $\tilde{\psi}_L(A, B, C, D)$ is Lyapunov-balanced with Lyapunov–Gramian Σ . The converse follows in the same way. Note that when restricted to the subclass of LQG-balanced systems, $\tilde{I}_{\psi_L} \circ \psi_L$ is indeed the identity map.

Thus, if $(A, B, C, D) \in L_n^{p,m}$ is LQG-balanced with LQG-Gramian Σ , $\phi_L(A, B, C, D)$ is also LQG-balanced with Gramian Σ . Moreover, if $(A, B, C, D) \sim (A_1, B_1, C_1, D_1)$, then $\tilde{\psi}_L(A, B, C, D) \sim \tilde{\psi}_L(A_1, B_1, C_1, D_1)$ and thus, since ϕ_D is a canonical form,

$$\phi_D \tilde{\psi}_L(A, B, C, D) = \phi_D \tilde{\psi}_L(A_1, B_1, C_1, D_1).$$

This proves $\phi_L(A, B, C, D) = \phi_L(A_1, B_1, C_1, D_1)$. Finally,

$$\phi_L(A, B, C, D) \sim \tilde{I}_{\psi_L} \tilde{\psi}_L(A, B, C, D) = (A, B, C, D)$$

shows that ϕ_L is indeed a canonical form. \square

In view of the above theorem, one obtains a canonical form for discrete-time LQG-balanced minimal systems via the corresponding canonical form for Lyapunov-balanced discrete-time asymptotically stable systems using the L -characteristic map. The balanced canonical form for $L_n^{p,m}$ is schematically represented by the following Fig. 1.

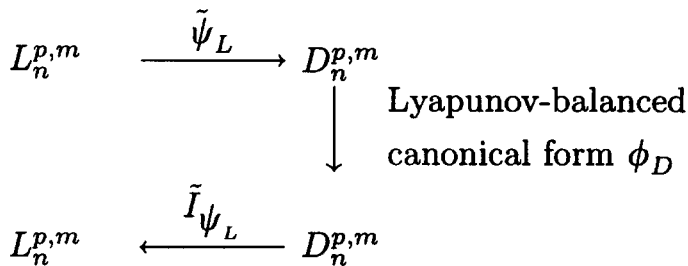


Fig. 1. Discrete-time LQG-balanced canonical form ϕ_L .

The main advantage of this procedure is the fact that it results in a construction of a *balanced* canonical form for arbitrary, not necessarily stable, discrete-time minimal systems. Other canonical forms for the similarity action on minimal systems (A, B, C) are well known, with important contributions by Kalman [8], Rissanen [9], Popov [10], and Bosgra and van der Weiden [11]. Most of these canonical forms for multivariable systems are generalizations of the observer or controller canonical forms for SISO systems. Their significance for system identification originates from the fact that there is a transparent relation between the parameters of the canonical form and the coefficients of the associated transfer function.

On the other hand, in a more recent work, Ober [12] has constructed several new canonical forms for state-space equivalence, based on balanced realizations. These canonical forms share the striking advantages of balancing with respect to system truncation, model reduction, robustness properties, and system identification. Furthermore, the geometrically very simple parameter space of balanced parametrizations gives a better understanding of basic topological aspects, such as the number of connected components of the manifold of fixed-degree systems, and a variety of diffeomorphisms between various system classes.

For the class of linear time-invariant minimal systems, the known balanced canonical form is based on Lyapunov-balancing, which is restricted to asymptotically stable systems. Ober and Fuhrmann [3] introduced a balanced canon-

ical form for continuous-time minimal systems with no spectral restrictions, via a characteristic map. The present paper provides the analogous construction for the discrete case.

6. Conclusions

In this paper, a Bucy relation is obtained for discrete-time minimal systems. Using this relation, a new bijection is constructed between $L_n^{p,m}$, the set of minimal systems with no restrictions on the pole location, and $D_n^{p,m}$, the set of discrete-time asymptotically stable minimal systems, via the characteristic map ψ_L and its inverse I_{ψ_L} . The characteristic map is given in terms of explicit state space formulas, and it applies to systems which are not necessarily strictly causal. It gives rise to a diffeomorphism between $L_{n,0}^{p,m}$ and $D_{n,0}^{p,m}$, the corresponding subsets of systems with zero feedthrough term, in $L_n^{p,m}$ and $D_n^{p,m}$, respectively. There are, of course, simpler ways to construct homeomorphisms and even diffeomorphisms between these spaces, for instance, by a mere scaling of the system matrices, see e.g. the work by Helmke [13] or Hanzon [14]. Although such a scaling leads indeed to canonical forms for the similarity action on $L_n^{p,m}$, it fails for the construction of *balanced* canonical forms. The characteristic map reveals close relations between the solutions of the corresponding algebraic Riccati and Lyapunov equations, which are used for the construction of our balanced canonical form for LQG-balanced minimal discrete-time systems.

Appendix A

Proof of Lemma 4.2. Since X is the stabilizing solution of the Riccati equation, it is clear that $\mathcal{A} = A - BF$ is stable. Moreover, since (A, B) is reachable, we also have reachability of $(A - BF, B)$ and hence of $(A - BF, B(R + B^*XB)^{-1/2})$. For checking observability, we write

$$\mathcal{A} = (I + YX)^{-1}(A - HC)(I + YX),$$

as established in (14). Thus $(\mathcal{A}, \mathcal{C})$ is similar to $(A - HC, (S + CYC^*)^{-1/2}C)$, and an analogous argument as above yields the observability of this matrix pair.

The stabilizing solutions of the ARE's for the system $(TAT^{-1}, TB, CT^{-1}, D)$, where T is nonsingular, are given by $T^{-1*}XT^{-1}$ and TYT^* , respectively. Hence we get

$$\psi_L \left(\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array} \right) =: \left(\begin{array}{c|c} \hat{\mathcal{A}} & \hat{\mathcal{B}} \\ \hline \hat{\mathcal{C}} & \hat{\mathcal{D}} \end{array} \right)$$

with $\hat{\mathcal{D}} = \mathcal{D}$,

$$\begin{aligned}
\hat{\mathcal{A}} &= TAT^{-1} - TB(R + B^*T^*(T^{-1*}XT^{-1})TB)^{-1} \\
&\quad (B^*T^*(T^{-1*}XT^{-1})TAT^{-1} + D^*CT^{-1}) = T\mathcal{A}T^{-1}, \\
\hat{\mathcal{B}} &= TB(R + B^*T^*(T^{-1*}XT^{-1})TB)^{-1/2} = T\mathcal{B}, \\
\hat{\mathcal{C}} &= (S + CT^{-1}(TYT^*)T^{-1*}C^*)^{-1/2}CT^{-1}(I + (TYT^*)(T^{-1*}XT^{-1})) \\
&= \mathcal{C}T^{-1}. \quad \square
\end{aligned}$$

Proof of Proposition 4.3. We prove that $\mathcal{A}(I + YX)^{-1}Y\mathcal{A}^* + \mathcal{B}\mathcal{B}^* = (I + YX)^{-1}Y$. First, we plug in $\mathcal{A} = A - BF = \bar{A} - \bar{B}\bar{F}$ and use the Bucy relation (19) to obtain

$$\begin{aligned}
\mathcal{A}(I + YX)^{-1}Y\mathcal{A}^* &= (\bar{A} - \bar{B}\bar{F})(I + YX)^{-1}Y(\bar{A} - \bar{B}\bar{F})^* \\
&= (I + YX)^{-1}(\bar{A} - \bar{H}\bar{C})Y(\bar{A}^* - \bar{F}^*\bar{B}^*).
\end{aligned}$$

Using the definition of \bar{F} , this results in

$$\mathcal{A}(I + YX)^{-1}Y\mathcal{A}^* = (I + YX)^{-1}(\bar{A} - \bar{H}\bar{C})Y\bar{A}^*(I - X\bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*).$$

Using (17), we can also write

$$\mathcal{A}(I + YX)^{-1}Y\mathcal{A}^* = (I + YX)^{-1}(Y - \bar{B}\bar{B}^*)(I - X\bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*),$$

or, with $P = (I + YX)^{-1}Y$,

$$P - \mathcal{A}P\mathcal{A}^* = (I + YX)^{-1}(\bar{B}\bar{B}^* + (Y - \bar{B}\bar{B}^*)X\bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*).$$

Thus, it remains to be proven that the right-hand side of the last equation equals $\mathcal{B}\mathcal{B}^*$. Note that

$$\mathcal{B}\mathcal{B}^* = B(R + B^*XB)^{-1}B^* = \bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*,$$

hence $\mathcal{B}\mathcal{B}^* = (I + YX)^{-1}(\bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^* + YX\bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*)$. The equality

$$\bar{B}\bar{B}^* - \bar{B}\bar{B}^*X\bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^* = \bar{B}(I + \bar{B}^*X\bar{B})^{-1}\bar{B}^*$$

can easily be verified and concludes the proof for the first Lyapunov equation. The second equation is treated analogously. Uniqueness of the solutions follows from the fact that \mathcal{A} is discrete-time asymptotically stable. \square

Proof of Proposition 4.5. We show that $V := (I + QP)^{-1}Q$ solves (24). The second equation is treated in the same way. The feedthrough terms \mathcal{Q} and D_1 are equal, and thus also the matrices R and S which are defined as usual. With $\bar{A}_1 = A_1 - B_1R^{-1}D_1^*C_1$, $\bar{B}_1 = B_1R^{-1/2}$, $\bar{C}_1 = S^{-1/2}C_1$,

$$\begin{aligned}\bar{A}_1 &= \mathcal{A} + \mathcal{B}(I - \mathcal{B}^*V\mathcal{B}^*)^{-1}\mathcal{B}^*V\mathcal{A} = (I - \mathcal{B}\mathcal{B}^*V)^{-1}\mathcal{A}, \\ \bar{B}_1 &= \mathcal{B}(I - \mathcal{B}^*V\mathcal{B}^*)^{-1/2}, \\ \bar{C}_1 &= (I - \mathcal{C}P(I + QP)^{-1}\mathcal{C}^*)^{-1/2}\mathcal{C}(I + PQ)^{-1}.\end{aligned}$$

Moreover,

$$\begin{aligned}\bar{A}_1 &= (I - \mathcal{B}\mathcal{B}^*V)^{-1}\mathcal{A} = (I - \mathcal{B}\mathcal{B}^*Q(I + PQ)^{-1})^{-1}\mathcal{A} \\ &= (I + PQ)(I + PQ - \mathcal{B}\mathcal{B}^*Q)^{-1}\mathcal{A}\end{aligned}$$

and, using the Lyapunov equations,

$$\begin{aligned}(I + PQ)^{-1}\bar{A}_1 &= (I + \mathcal{A}P\mathcal{A}^*Q)^{-1}\mathcal{A} = \mathcal{A}(I + P\mathcal{A}^*Q\mathcal{A})^{-1} \\ &= \mathcal{A}(I + PQ - P\mathcal{C}^*\mathcal{C})^{-1}.\end{aligned}$$

Writing (24) in standard form, what we need to prove is

$$V - \bar{C}_1^*\bar{C}_1 \stackrel{!}{=} \bar{A}_1^*V\bar{A}_1 - \bar{A}_1^*V\bar{B}_1(I + \bar{B}_1^*V\bar{B}_1)^{-1}\bar{B}_1^*V\bar{A}_1. \quad (\text{A.1})$$

Since $I + \bar{B}_1^*V\bar{B}_1 = (I - \mathcal{B}^*V\mathcal{B}^*)^{-1}$, we have $\bar{B}_1(I + \bar{B}_1^*V\bar{B}_1)^{-1}\bar{B}_1^* = \mathcal{B}\mathcal{B}^*$, i.e., the right-hand side of the transformed ARE (A.1) is

$$\begin{aligned}\bar{A}_1^*(V - V\bar{B}_1(I + \bar{B}_1^*V\bar{B}_1)^{-1}\bar{B}_1^*V)\bar{A}_1 \\ &= \bar{A}_1^*(V - V\mathcal{B}\mathcal{B}^*V)\bar{A}_1 = \bar{A}_1^*V(I - \mathcal{B}\mathcal{B}^*V)\bar{A}_1 = \bar{A}_1^*V\mathcal{A} \\ &= (I + QP - \mathcal{C}^*\mathcal{C}P)^{-1}\mathcal{A}^*(I + QP)V\mathcal{A} = (I + QP - \mathcal{C}^*\mathcal{C}P)^{-1}\mathcal{A}^*Q\mathcal{A} \\ &= (I + QP - \mathcal{C}^*\mathcal{C}P)^{-1}(Q - \mathcal{C}^*\mathcal{C}).\end{aligned} \quad (\text{A.2})$$

The left-hand side of (A.1) transforms to

$$\begin{aligned}V - \bar{C}_1^*\bar{C}_1 \\ &= (I + QP)^{-1}Q - (I + QP)^{-1}\mathcal{C}^*(I - \mathcal{C}P(I + QP)^{-1}\mathcal{C}^*)^{-1}\mathcal{C}(I + PQ)^{-1} \\ &= (I + QP)^{-1}Q - (I + QP)^{-1}(I - \mathcal{C}^*\mathcal{C}P(I + QP)^{-1})^{-1}\mathcal{C}^*\mathcal{C}(I + PQ)^{-1} \\ &= (I + QP)^{-1}Q - (I + QP - \mathcal{C}^*\mathcal{C}P)^{-1}\mathcal{C}^*\mathcal{C}(I + PQ)^{-1}.\end{aligned} \quad (\text{A.3})$$

Thus, it remains to be shown that (A.2) and (A.3) coincide. This follows from the trivial equality

$$(I + QP - \mathcal{C}^*\mathcal{C}P)^{-1}(I + QP - \mathcal{C}^*\mathcal{C}P)Q = Q = (I + QP)^{-1}Q(I + PQ)$$

by subtracting $(I + QP - \mathcal{C}^*\mathcal{C}P)^{-1}\mathcal{C}^*\mathcal{C}$ from both sides and multiplying the resulting equation

$$\begin{aligned}
& (I + QP - \mathcal{C}^* \mathcal{C} P)^{-1} (Q - \mathcal{C}^* \mathcal{C}) (I + PQ) \\
&= (I + QP)^{-1} Q (I + PQ) - (I + QP - \mathcal{C}^* \mathcal{C} P)^{-1} \mathcal{C}^* \mathcal{C}
\end{aligned}$$

by $(I + PQ)^{-1}$ from the right. The closed loop matrix associated to (24) is

$$\begin{aligned}
A_1 - B_1 F_1 &:= \bar{A}_1 - \bar{B}_1 \bar{F}_1 := \bar{A}_1 - \bar{B}_1 (I + \bar{B}_1^* V \bar{B}_1)^{-1} \bar{B}_1^* V \bar{A}_1 \\
&= (I - \mathcal{B} \mathcal{B}^* V) \bar{A}_1 = \mathcal{A}.
\end{aligned}$$

In view of the asymptotic stability of \mathcal{A} , this shows that V is indeed a stabilizing solution, and thus uniquely determined. \square

Proof of Proposition 4.6. Let (A, B, C, D) be a minimal system, and let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ be its characteristic system; furthermore, let X and Y be the stabilizing solutions to the associated Riccati equations. In Proposition 4.3 we have shown that the solutions of the Lyapunov equations associated to $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ are given by $P = Y(I + XY)^{-1}$ and $Q = X(I + YX)$. We use the notation:

$$I_{\psi_L} \left(\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right) = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right).$$

The determination of the feedthrough term is easy; we see that $D_1 = \mathcal{D} = D$. Next we show that $B_1 = B$ using the following formulas:

$$\begin{aligned}
PQ &= Y(I + XY)^{-1} (I + XY) X = YX, \quad QP = XY, \\
(I + QP)^{-1} Q &= (I + XY)^{-1} (I + XY) X = X, \\
P(I + QP) &= Y(I + XY)^{-1} (I + XY) = Y.
\end{aligned}$$

Rewrite \mathcal{B} as

$$\mathcal{B} = B(R + B^* X B)^{-1/2} = \bar{B}(I + \bar{B}^* X \bar{B})^{-1/2}.$$

Then, from the definition of B_1 ,

$$\begin{aligned}
B_1 R^{-1/2} &= \mathcal{B}(I - \mathcal{B}^* (I + QP)^{-1} Q \mathcal{B})^{-1/2} = \mathcal{B}(I - \mathcal{B}^* X \mathcal{B})^{-1/2} \\
&= \bar{B}(I + \bar{B}^* X \bar{B})^{-1/2} \left(I - (I + \bar{B}^* X \bar{B})^{-1/2} \bar{B}^* X \bar{B} (I + \bar{B}^* X \bar{B})^{-1/2} \right)^{-1/2} \\
&= \bar{B}(I + \bar{B}^* X \bar{B})^{-1/2} \left((I + \bar{B}^* X \bar{B})^{-1} \right)^{-1/2} = \bar{B}.
\end{aligned}$$

Thus, in view of $\bar{B} = BR^{-1/2}$, we obtain $B_1 = B$. Analogously, we can write

$$\mathcal{C} = (I + \bar{C} Y \bar{C}^*)^{-1/2} \bar{C} (I + YX),$$

and thus, using the definition of C_1 ,

$$\begin{aligned}
S^{-1/2}C_1(I + PQ) &= (I - \mathcal{C}P(I + QP)^{-1}\mathcal{C}^*)^{-1/2}\mathcal{C} \\
&= (I - \mathcal{C}(I + YX)^{-1}Y(I + XY)^{-1}\mathcal{C}^*)^{-1/2}\mathcal{C} \\
&= (I - (I + \bar{C}Y\bar{C}^*)^{-1/2}\bar{C}Y\bar{C}^*(I + \bar{C}Y\bar{C}^*)^{-1/2})^{-1/2}\mathcal{C} \\
&= (I + \bar{C}Y\bar{C}^*)^{1/2}\mathcal{C} = \bar{C}(I + YX).
\end{aligned}$$

Hence $S^{-1/2}C_1 = \bar{C} = S^{-1/2}C$, i.e., $C_1 = C$. Finally, we calculate A_1 . From (12),

$$\mathcal{A} = \bar{A} - \bar{B}\bar{F} = (I + \bar{B}\bar{B}^*X)^{-1}\bar{A}.$$

Using $\bar{B} = \mathcal{B}(I - \mathcal{B}^*X\mathcal{B})^{-1/2}$ from above, we obtain

$$\begin{aligned}
\bar{A} &= (I + \bar{B}\bar{B}^*X)\mathcal{A} = (I + \mathcal{B}(I - \mathcal{B}^*X\mathcal{B})^{-1}\mathcal{B}^*X)\mathcal{A} \\
&= \mathcal{A} + \mathcal{B}(I - \mathcal{B}^*(I + QP)^{-1}Q\mathcal{B})^{-1}\mathcal{B}^*(I + QP)^{-1}Q\mathcal{A}.
\end{aligned} \tag{A.4}$$

Moreover, since

$$A = \bar{A} + BR^{-1}D^*C = \bar{A} + B_1R^{-1}D^*C_1,$$

a direct comparison to the definition of A_1 together with (A.4) yields $A_1 = A$. \square

Proof of Proposition 4.7. Let $\Sigma = (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in D_n^{p,m}$ and let $\Sigma_1 = (A_1, B_1, C_1, D_1) = I_{\psi_L}(\Sigma)$. We need to prove that $\psi_L(\Sigma_1) =: (\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1) = \Sigma$. The solutions of the ARE's associated to Σ_1 are denoted by X and Y , and the solutions of the Lyapunov equations for Σ are P and Q . The feedthrough terms are all equal. With $\bar{B}_1 = B_1R^{-1/2}$,

$$\mathcal{B}_1 = B_1(R + B_1^*XB_1)^{-1/2} = \bar{B}_1(I + \bar{B}_1^*X\bar{B}_1)^{-1/2}.$$

Since

$$\bar{B}_1 = \mathcal{B}(I - \mathcal{B}^*(I + QP)^{-1}Q\mathcal{B})^{-1/2} = \mathcal{B}(I - \mathcal{B}^*X\mathcal{B})^{-1/2}$$

and

$$\begin{aligned}
I + \bar{B}_1^*X\bar{B}_1 &= I + (I - \mathcal{B}^*X\mathcal{B})^{-1/2}\mathcal{B}^*X\mathcal{B}(I - \mathcal{B}^*X\mathcal{B})^{-1/2} \\
&= (I - \mathcal{B}^*X\mathcal{B})^{-1/2}(I - \mathcal{B}^*X\mathcal{B} + \mathcal{B}^*X\mathcal{B})(I - \mathcal{B}^*X\mathcal{B})^{-1/2} \\
&= (I - \mathcal{B}^*X\mathcal{B})^{-1},
\end{aligned}$$

we obtain

$$\mathcal{B}_1 = \mathcal{B}(I - \mathcal{B}^*X\mathcal{B})^{-1/2}(I - \mathcal{B}^*X\mathcal{B})^{1/2} = \mathcal{B}.$$

Similarly, with $\bar{C}_1 = S^{-1/2}C_1$,

$$\mathcal{C}_1 = (S + C_1YC_1^*)^{-1/2}C_1(I + YX) = (I + \bar{C}_1Y\bar{C}_1^*)^{-1/2}\bar{C}_1(I + YX)$$

and

$$\bar{C}_1 = (I - \mathcal{C}P(I + QP)^{-1}\mathcal{C}^*)^{-1/2}\mathcal{C}(I + PQ)^{-1}.$$

Thus, noting that $P(I + QP)^{-1} = (I + PQ)^{-1}P = (I + PQ)^{-1}Y(I + QP)^{-1}$,

$$I + \bar{C}_1 Y \bar{C}_1^* = (I - \mathcal{C}P(I + QP)^{-1}\mathcal{C}^*)^{-1}.$$

Thus

$$\mathcal{C}_1 = \mathcal{C}(I + PQ)^{-1}(I + YX) = \mathcal{C}.$$

Finally, let $\bar{A}_1 = A_1 - B_1 R^{-1} D_1^* C_1$, then

$$\mathcal{A}_1 = (I + \bar{B}_1 \bar{B}_1^* X)^{-1} \bar{A}_1$$

and

$$\bar{A}_1 = \mathcal{A} + \mathcal{B}(I - \mathcal{B}^* X \mathcal{B})^{-1} \mathcal{B}^* X \mathcal{A} = \mathcal{A} + \bar{B}_1 \bar{B}_1^* X \mathcal{A},$$

thus $\mathcal{A}_1 = \mathcal{A}$. \square

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